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Fiber Braids and Knots

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Fiber braids and knots are fiber bundles having braids or knots as base space. The use of these in concrete physical problems is sketched and the relevant topological classification is discussed.

INTRODUCTION AND MOTIVATION

In a previous note (Gaeta, 1992) we remarked how knot theory and knot invariants (Burde and Zieschang, 1986; Kauffman, 1983, 1987*a*, 1988, 1990*a*; Rolfsen, 1976) can be useful also in connection with very simple classical mechanics problems—besides the known applications to modern theoretical physics issues (Yang, 1967; Yang and Ge, 1989; Jimbo *et al.*, 1989; Wadati *et al.*, 1989; Kauffman, 1987*b*, 1990*b*, 1991; Lusanna, 1990)—such as the study of a nonlinear oscillator with damping and periodic external forcing.

In the present note, we introduce an extension of knots and braids which turns out to be quite a natural tool in the topological study of closed trajectories in fluid mechanics and magnetohydrodynamics, as well as in the study of the dynamics of point particles carrying an abstract field.

We begin by mentioning a couple of simple physical systems for which we would be interested in a topological classification of periodic motions.

A. Consider a point particle on the line, obeying the equations of a damped, periodically forced, nonlinear oscillator and carrying a field A, valued in a manifold $M \subseteq \mathbf{R}^N$; this field obeys a dynamics which depends on the position of the point particle. The equations of motions will then be

$$\ddot{x} = F(x, \dot{x}, t), \qquad \dot{A} = \phi(A, x, t) \tag{1}$$

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(As an example, one can think of an electron in a crystal, constrained to linear motions, in a magnetic field.) Can we classify topologically the periodic solutions for both x and A?

B. A charged particle moving in a purely magnetic field has constant speed; once its initial speed is assigned, we can consider its trajectories. Closed trajectories obviously correspond to knots. Suppose we know one of these trajectories K_0, and that neighboring particles move without leaving an invariant tube T_0 centered on K_0. Can we topologically classify the collective motions of the charged plasma inside T_0?

In order to answer these kinds of questions, we introduce the concept of *fiber braid*, which is actually an immediate extension of the more familiar "ribbon braids" (see, e.g., Reshetikhin, 1989).

We will consider a fiber bundle (E, B, π) with as base B a knot K_0, and fiber $\pi^{-1}(x) = M$. Let us see how this can be of use for the cases considered above.

A. For the correspondence between periodic solutions for the motion of the particle and knots in the space $\mathbf{R}^2 \times S^1$, see Gaeta (1992). It is clear that, as $x(t)$ moves in $\mathbf{R}$, the field $A(t)$ performs a motion in the manifold M, which is indeed the fiber of the fiber knot; if $A(t)$ is periodic with the same period $T = T_0$ as the particle motion, this trajectory must close once $x(t)$ has gone around the knot K_0. The topological classification of periodic motions relies therefore on the description of the knot corresponding to the point particle motion, and on the classification of closed paths in M, i.e., on $\pi(M)$.[2] Remark that if we do not have $T = T_0$ the situation is slightly different; in order to have periodic solutions we must impose that after $x(t)$ has described some number n of loops around K_0, also the trajectory for $A(t)$ closes: in this case we should use $\mathbf{Z} \times \pi(M)$ rather than $\pi(M)$ alone.

B. Consider a cross section of the flow tube T_0 as the fiber of our bundle: this gives the two-disk D^2; notice that no trajectory can pass through points of K_0 unless it coincides with K_0, so that the relevant manifold M is actually the punctured disk $D_0^2 \equiv D^2 \backslash \{0\}$. The trajectories of particles in the flow tube can be described as $\phi(x)$, where $x \in K_0$ and $\phi(x) \in \pi^{-1}(x)$. Since D_0^2 is contractible to the circle S^1, we could actually deal with a circle bundle, $M = S^1$.

If we are interested in the trajectory of a single particle evolving in T_0, the situation is pretty much the same as in case A, but since we are asking about periodic motions of all particles, we should not look at a single section of the bundle, but rather see the trajectories as motions following a connection defined in the bundle. This leads to classifying connections ω that,

[2]No confusion should be possible between the projection of the bundle and the fundamental group of the manifold, both denoted by π.

integrated over the loop K_0, give the identity map. In other words, we should look at the *holonomy group* of the bundle (Dubrovin *et al.*, 1982).

BRAIDS

Let us now consider braids;[3] the most general braid is generated by the elements sketched in Figure 1 by means of two operations, which we will call here composition or product, denoted as $\circ$, and juxtaposition or sum, denoted by $\oplus$; examples of these are given in Figure 2.

The number of strings in the braid is called its dimension (or order); the braids of dimension n form a group under the product, called the braid group or order n, B_n; clearly we have

$$\circ: \quad B_n \times B_n \to B_n; \qquad \oplus: \quad B_n \times B_m \to B_{n+m} \tag{2}$$

The Reidemeister moves *RII* and *RIII* are readily expressed in terms of relations in the braid group:

$$RII \Leftrightarrow \sigma \circ \sigma^{(-1)} = e \oplus e = \sigma^{(-1)} \circ \sigma \tag{3}$$

$$RIII \Leftrightarrow (\sigma \oplus e) \circ (e \oplus \sigma) \circ (\sigma \oplus e) = (e \oplus \sigma) \circ (\sigma \oplus e) \circ (e \oplus \sigma) \tag{4}$$

The expression of the Reidemeister move *RIII* is nothing else than the Yang–Baxter equation.[4]

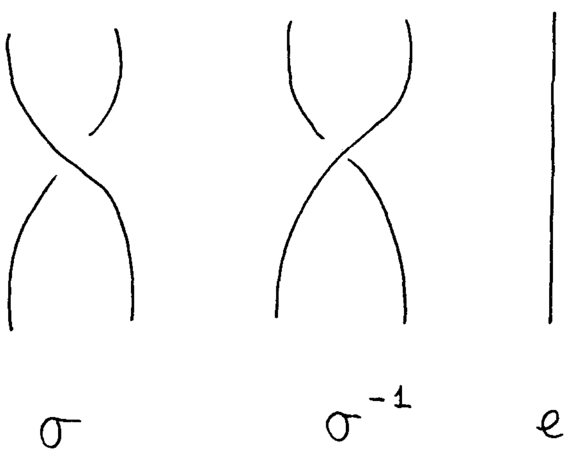

Fig. 1. Generators of braids; these elements generate the most general braid, as explained in the text.

[3]As is well known, every knot or link can be encoded as a braid of appropriate dimension by a canonical procedure.

[4]This is often encountered in the notation $R_{12} \equiv (\sigma \oplus e)$, $R_{23} \equiv (e \oplus \sigma)$.

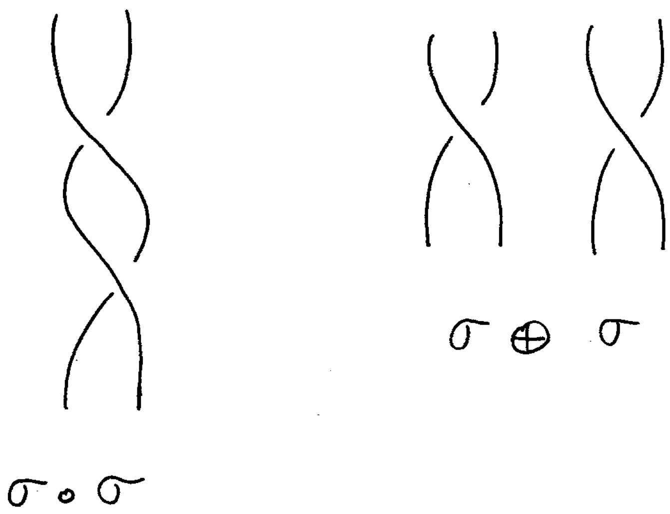

Fig. 2. An example of braids generated by the elements of Figure 1 by means of the two basic operations of product and sum.

RIBBON AND FIBER BRAIDS

A meaningful generalization of braids is given by the so-called "ribbon braids;" the direct analogue of the generators of Figure 1 are given in Figure 3, together with two new generators that must now be added (θ, θ^{-1} in Figure 3). Notice that for ribbon knots a Reidemeister move *RI* does generate a θ-type element; see Figure 4.

It requires very little abstraction to see $\{\theta, \theta^{-1}\}$ as generating the homotopy group of S^1, i.e., $\mathbf{Z}$: indeed, a ribbon can also be seen as a framing in a tube with axis on one of the borders of the ribbon (this applies immediately to our fluid example).

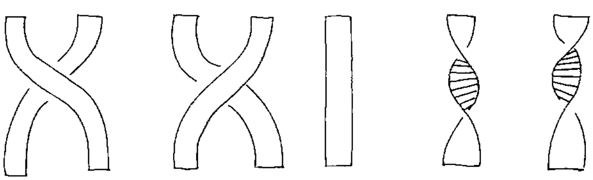

Fig. 3. Generators for ribbon braids. Together with elements corresponding to ordinary braid generators, new generators are present.

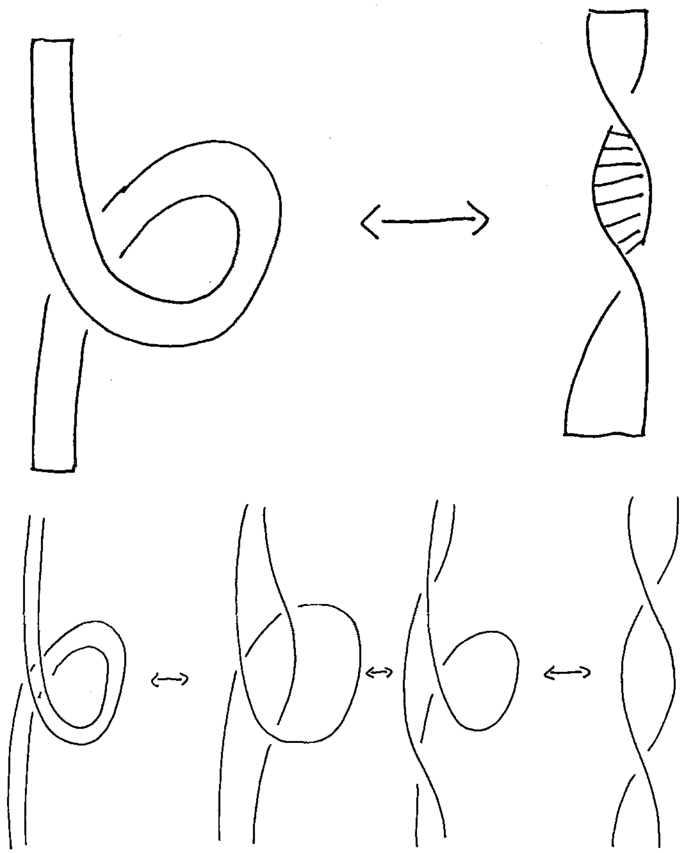

Fig. 4. The effect of a Reidemeister move *RI* on ribbon braids. This can be seen directly on the ribbon (a), or considering a ribbon braid as equivalent to the braid given by the bords of the ribbon (b).

The relations involving only $\{\sigma, \sigma^{-1}, e\}$ continue to satisfy (3), (4); as for the new elements, they obviously satisfy $\theta \circ \theta^{-1} = \theta^{-1} \circ \theta = e$, as well as $(\alpha, \beta = \pm 1)$

$$\begin{aligned} \sigma^{\alpha} \circ (e \oplus \theta^{\beta}) &= (\theta^{\beta} \oplus e) \circ \sigma^{\alpha} \\ \sigma^{\alpha} \circ (\theta^{\beta} \oplus e) &= (e \oplus \theta^{\beta}) \circ \sigma^{\alpha} \end{aligned} \tag{5}$$

Let us, for definiteness, fix the dimension n of the braids. It is clear that $\{\sigma, \sigma^{-1}, e\}$ still generate B_n, while $\{\theta, \theta^{-1}, e\}$ generate $\mathbf{Z}^n$. By the relations (5), which define the interaction of the two kinds of generators, it is clear that the total group these generate is $B_n \otimes_{\rightarrow} \mathbf{Z}^n$ (semidirect product, with B_n acting on $\mathbf{Z}^n$).

This setting is readily generalized to the required degree of abstraction: let G denote the relevant group of transformations (possibly, but not necessarily, Abelian) of the fiber M [depending on the physical problem at hand, as we have seen, this will be $\pi(M)$ or the holonomy group of a fiber bundle over S^1 with fiber M—from now on denoted $h(M)$], and let $\{\theta_i^\alpha;\ i=1,\ldots,n;\ \alpha=\pm 1\}$ be its generators.

As in the ribbon case—fixing again the dimension n of the braids for definiteness—we have that $\{\sigma, \sigma^{-1}, e\}$ generate the group B_n, while $\{\theta_i, \theta_i^{-1}, e\}$ generate the group $G^n \equiv (G \times \cdots \times G)$.

If we assume the relations $(\alpha, \beta = \pm 1)$

$$\begin{aligned} \sigma^\alpha \circ (e \oplus \theta_i^\beta) &= (\theta_i^\beta \oplus e) \circ \sigma^\alpha \\ \sigma^\alpha \circ (\theta_i^\beta \oplus e) &= (e \oplus \theta_i^\beta) \circ \sigma^\alpha \end{aligned} \tag{6}$$

to be satisfied for all i, then we have that the total group is

$$B_n^{(M)} = B_n \otimes_{\rightarrow} G^n \tag{7}$$

This also provides a topological classification of fiber braids.

DISCUSSION

It is worth spending a few words on the relations (5), (6). These essentially say that twisting of the fiber does not interact with the braid structure; indeed, a graphical translation of (5) is provided by Figure 5 and shows this to be quite a reasonable assumption.

Therefore, until there is no physical interaction among the two branches of a crossing, relations (5), (6) are perfectly justified; obviously, they just claim topological equivalence, while if we consider fluid motion and attach an energy functional to it, this will not in general depend only on the topology (Moffat, 1990).

We also remark that, among different physical applications, braids can also be seen as representing world lines of scattering particles [the Yang–Baxter equation represents then the factorization of the S matrix into two-particle scatterings (Kauffman, 1990*b*, 1991)]. In this setting it would be quite natural to consider also internal degrees of freedom of the particles, in the sense of gauge theories; these are indeed described in terms of fiber bundles, so that our concept of fiber braids is again quite pertinent. Notice that in this case (6) amounts to saying that scattering does not affect the internal state of the particle; if this does not apply, relation (6) should be replaced by more complicated ones, which it would be out of place to discuss here.

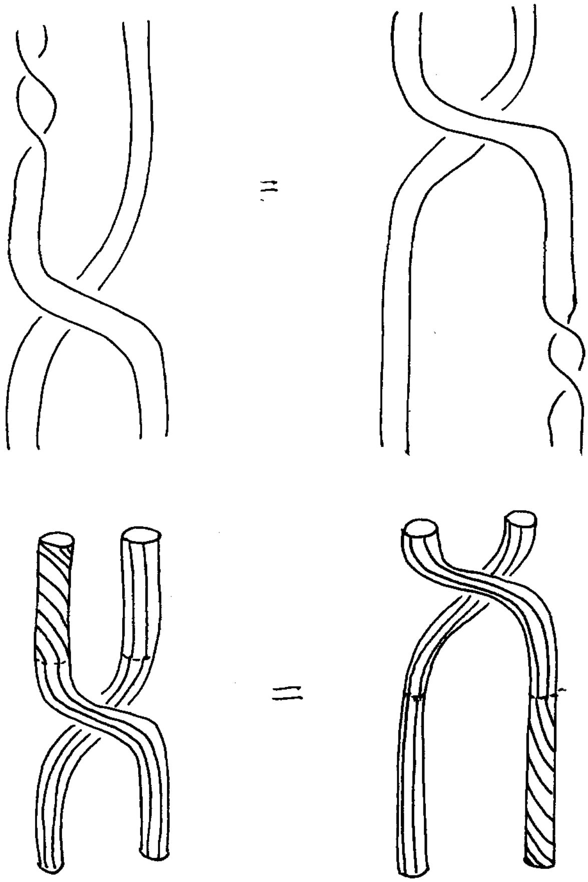

Fig. 5. The twisting of ribbons commutes with the crossings. This gives a justification of equations (5), (6). (a) A ribbon braid; (b) the ribbon as identifying a frame in a cylinder, or a connection in a flow tube.

FIBER KNOTS

We have shown that fiber braids are classified by the group (7); the physical problems providing the motivation for our investigation were set in terms of knots, and braids were introduced as a practical way to study and encode knots. We should therefore check how the situation changes when we "close the braid," i.e., when we pass from a braid to the corresponding knot K.

In Figure 6 we have depicted a fiber braid by drawing the diagram corresponding to its base braid (the one corresponding to the threefoil knot), and by the circle and the square we denote symbolically two elements of the

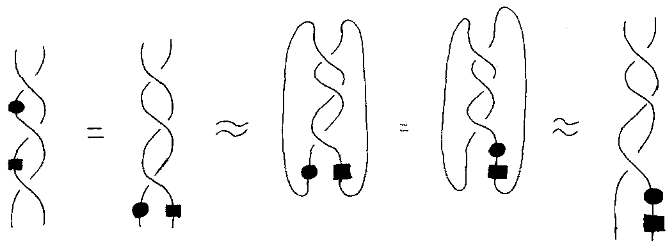

Fig. 6. By using the relations (5), (6), we can always concentrate the nontrivial actions on the fiber in a specific region of the fiber knot; see text for details.

group G; the $=$ sign denotes topological equivalence [including the possibility affirmed by (6) to push elements of G along the strings], and $\simeq$ denotes the standard correspondence between braids and knots.

A little reflection shows that—assuming the physical validity of (6)—the message of Figure 6 is valid in general: on a knot we can group the actions on the fiber in a specific region[5] so that in the braid representation of the knot only one of the strings will be affected by nontrivial action of G.

More formally, a knot is a cyclic $\circ$-product of elements of the form $b^{(i)} \in B_n$, which are generated from $\{\sigma, \sigma^{(-1)}, \theta_i, \theta_i^{(-1)}, e\}$ by $\oplus$ alone, so that by repeated application of (6) we can arrive at the situation considered above.[6]

Therefore, the topological classification of fiber knots is given by $B \oplus_{\rightarrow} G$ (and not $B \oplus_{\rightarrow} G^n$), where B is the braid group.

Notice that now B acts on G in a nonsimple way: it essentially tells, given a "horizontal sequence" $\alpha = (\theta_{i_1} \oplus \cdots \oplus \theta_{i_n})$, how this is transformed into a "vertical sequence" $\beta = (e \oplus \cdots \oplus e \oplus \tilde{\alpha})$, $\tilde{\alpha} = (\theta_{i_{\pi(1)}} \circ \cdots \circ \theta_{i_{\pi(n)}})$, by pushing the θ_i along the knot (see the third equivalence of Figure 6).

If G is Abelian, the action of B on G is actually trivial, and the fiber knot is identified by a pair $(b, g) \in (B, G)$, but in general G could be non-Abelian. This can be the case, e.g., if $G = \pi(W)$, with W the orbit space $\Omega(M, \mathscr{G})$ of a Lie group $\mathscr{G}$ on a manifold M [a non-Abelian $\pi(W)$ corresponds to the appearance of orbifolds].

EXAMPLES

We would now like to look again at the physical situations mentioned in the Introduction as motivating examples.

[5]This amounts to saying that for fiber bundles over $S^1 = [0, 2\pi]$ we can concentrate the relevant Dehn surgeries in a small interval, say $[0, \varepsilon]$.

[6]Notice that in the same way, for an m-component link we can concentrate nontrivial elements of G on m strings only of the corresponding braid.

A. In this case the relevant group is $G=\pi(M)$, so that periodic solutions $(x(t), A(t))$ are topologically classified by an integer n (the period being $n\tau$, with τ the period of the forcing term) and by elements of $B_n \oplus_{\rightarrow} \pi(M)$.

For example, if the field A represents a magnetic moment of the particle, of constant amplitude but free to point along any direction of three-dimensional space, we have $M=S^2$, while if it can point only in directions orthogonal to the line along which the particle moves, we have $M=S^1$. Notice that in the first case $\pi(S^2)=\{e\}$, so that no new topological information is provided by our setting, while in the second case $\pi(S^1)=\mathbf{Z}$, and the topological classification provided by considering the fiber knots is finer than the one provided by considering the knots (i.e., the particle trajectory) alone.

B. Here we are concerned with collective motions: as already pointed out, we have $M=D^2\backslash\{0\}$ and G the holonomy group of the M bundle over S^1, denoted as $G=h(M)$. The collective fluid motions of the kind considered here are classified by $B_n \oplus_{\rightarrow} h(M)$.

Here again, $h(M)=\mathbf{Z}$, so that considering fiber knots instead of knots amounts to considering, besides the topology of the reference trajectory, the helical twist (with respect to this) of neighboring trajectories.[7]

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[7]For the role of helicity in fluid mechanics see Moffat (1983, 1990, 1992), Moffat and Tsinober (1990), and Arnold and Khesin (1992).

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